Strong and Weak Discrete Maximum Principles for Matrices Associated with Elliptic Problems

Kazuo Ishihara
Department of Mathematics
Kyushu Institute of Technology
Tobata, Kitakyushu 804, Japan

In memory of James H. Wilkinson

Submitted by F. Chatelin

ABSTRACT

We consider the strong and weak discrete maximum principles for matrix equations associated with the elliptic problems. We also give some examples and an application to illustrate the usefulness of the discrete maximum principles.

1. INTRODUCTION

In the theory and applications of a wide class of real linear second order elliptic partial differential equations, the maximum principles play a basic role [8, 16, 17]. Let Ω be a bounded domain in the real m-dimensional Euclidean space $\mathbf{R}^m$, with boundary Γ. The second order elliptic partial differential operator $\mathscr{L}$ takes the form

$$\mathscr{L}u(x) \equiv -\sum_{i,j=1}^{m} \alpha_{i,j}(x)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{m} \beta_i(x)\frac{\partial u}{\partial x_i} + c_0(x)u(x),$$

where $\alpha_{i,j}(x)$, $\beta_i(x)$, $1 \leqslant i, j \leqslant m$, $c_0(x)$ are continuous in $\overline{\Omega} \equiv \Omega \cup \Gamma$; $c_0(x) \geqslant 0$; $\alpha_{i,j}(x) = \alpha_{j,i}(x)$, $1 \leqslant i, j \leqslant m$; and there exists a positive constant δ_0 such that

$$\sum_{i,j=1}^{m} \alpha_{i,j}(x)\xi_i\xi_j \geqslant \delta_0 \sum_{i=1}^{m} \xi_i^2 \qquad \text{in } \Omega$$

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for all $(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbf{R}^m$. Let $\partial/\partial\nu \equiv \sum_{i=1}^m \nu_i\, \partial/\partial x_i$ denote the outward directional derivative on the boundary in the direction $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_m)$ such that $\boldsymbol{\nu} \cdot \mathbf{n} > 0$, $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ being the unit outward normal vector on the boundary. Then, the strong and weak maximum principles are stated as the following results [8, Chapter 3; 16, Chapter 2]:

THEOREM 1 (Strong maximum principle). *Suppose that $u(x)$ satisfies*

$$\mathscr{L}u \leqslant 0 \qquad \text{in } \Omega.$$

If u attains a nonnegative maximum G at an interior point of Ω, then,

$$u \equiv G \qquad \text{in } \bar{\Omega}.$$

THEOREM 2 (Weak maximum principle). *Suppose that $u(x)$ satisfies*

$$\mathscr{L}u \leqslant 0 \qquad \text{in } \Omega.$$

Then,

$$\max_{x \in \bar{\Omega}} u(x) \leqslant \max\left\{0, \max_{x \in \Gamma} u(x)\right\}.$$

THEOREM 3. *Let $u(x)$ satisfy*

$$\mathscr{L}u \leqslant 0 \qquad \text{in } \Omega.$$

Suppose that u attains a nonnegative maximum G at a boundary point P. If P lies on the boundary of a ball in Ω, then

$$\frac{\partial u}{\partial \nu} > 0 \qquad \text{at } P$$

unless $u \equiv G$ in $\bar{\Omega}$.

REMARK 1. In the case where $\mathscr{L}u \equiv \sum_{i,j=1}^m \alpha_{i,j}(x)\partial^2 u/\partial x_i\, \partial x_j + \sum_{i=1}^m \beta_i(x)\partial u/\partial x_i$, *a nonnegative maximum G* in Theorems 1 and 3 is replaced by *a maximum G*. Also the conclusion in Theorem 2 is replaced by $\max_{x \in \bar{\Omega}} u(x) \leqslant \max_{x \in \Gamma} u(x)$.

REMARK 2. The condition that P lies on the boundary of a ball in Ω in Theorem 3 is called the *interior sphere property* at P of Γ. If this property is not satisfied, the conclusion of Theorem 3 will in general not be valid [17, p. 18].

REMARK 3. Replacing u by $-u$ in Theorems 1–3, we obtain the *minimum principles*.

In view of applications to numerical analysis, the discrete maximum principles are particularly useful in the resulting matrix equations, which approximate elliptic boundary value problems by employing the finite difference method or the finite element method. The weak discrete maximum principle, which is the discrete counterpart of Theorem 2, is well established [1–4, 18]. Moreover, this is applied not only to linear boundary value problems, but also to nonlinear boundary value problems [11, 12]. However, it seems to the author that in particular the discrete analogue of Theorem 3 is not yet established.

The objective of this paper is to show the strong and weak discrete maximum principles and related results. Further, some examples are also presented to illustrate their usefulness. Finally, we give an application of the discrete maximum principles to the radiation cooling problem with nonlinear boundary conditions.

2. DISCRETE MAXIMUM PRINCIPLES

The discrete equations for the approximation of the elliptic boundary value problem are written in matrix form. First, we present some matrix notation [19, pp. 18–23]. Given a matrix $A=(a_{i,j})$, $1 \leqslant i, j \leqslant N$, we say that A is reducible if there exists a permutation matrix Q such that

$$QAQ^{\mathrm{t}} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where t means the transpose, $A_{1,1}$ is an $r \times r$ submatrix, and $A_{2,2}$ is an $(N-r)\times(N-r)$ submatrix with $1 \leqslant r < N$. If no such permutation matrix exists, then A is irreducible. We say that A is diagonally dominant if

$$|a_{i,i}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{i,j}|, \qquad 1 \leqslant i \leqslant N.$$

Furthermore, A is irreducibly diagonally dominant if it is irreducible and diagonally dominant and the following strict inequality holds for at least one i_0 $(1 \leqslant i_0 \leqslant N)$:

$$|a_{i_0, i_0}| > \sum_{\substack{j=1 \\ j \neq i_0}}^{N} |a_{i_0, j}|.$$

In order to derive the discrete maximum principles, we prepare a useful lemma on the irreducibility of a matrix.

LEMMA 1 [15, p. 103]. *An $N \times N$ matrix $A = (a_{i,j})$ is irreducible if and only if for any two distinct indices $1 \leqslant i, j \leqslant N$, there exists a sequence of nonzero elements of A of the form*

$$\{a_{i, i_1}, a_{i_1, i_2}, \dots, a_{i_s, j}\},$$

where $i, i_1, i_2, \dots, i_s, j$ are distinct.

Consider the discrete approximation to the boundary value problem

$$\mathscr{L}u(x) = f(x) \qquad \text{in } \Omega, \tag{1}$$

$$\mathscr{B}u(x) = g(x) \qquad \text{on } \Gamma. \tag{2}$$

Here $\mathscr{B}$ is the boundary operator; for example,

$$\mathscr{B}u(x) \equiv \theta_1(x)\frac{\partial u}{\partial \mathbf{n}} + \theta_2(x)u.$$

By P_i, $1 \leqslant i \leqslant N$ (or P_i, $N+1 \leqslant i \leqslant N+M$) we denote the nodal points which belong to Ω (or Γ). The finite difference method or the finite element method with mesh size h leads to the following equations which approximate (1), (2):

$$\sum_{j=1}^{N} a_{i,j}w_j + \sum_{j=N+1}^{N+M} a_{i,j}w_j = f_i, \qquad 1 \leqslant i \leqslant N, \tag{3}$$

$$\sum_{j=1}^{N} a_{i,j}w_j + \sum_{j=N+1}^{N+M} a_{i,j}w_j = g_i, \qquad N+1 \leqslant i \leqslant N+M. \tag{4}$$

Here $(w_1, w_2, \ldots, w_{N+M}), (f_1, f_2, \ldots, f_N), (g_{N+1}, g_{N+2}, \ldots, g_{N+M})$ are approximate values of $(u(P_1), u(P_2), \ldots, u(P_{N+M}))$, $(f(P_1), f(P_2), \ldots, f(P_N))$, $(g(P_{N+1}), g(P_{N+2}), \ldots, g(P_{N+M}))$, respectively. The matrices $(a_{i,j})$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N+M$, and $(a_{i,j})$, $N+1 \leqslant i \leqslant N+M, 1 \leqslant j \leqslant N+M$, are the approximations to the operators $\mathscr{L}$ and $\mathscr{B}$, respectively. Define sets as follows:

$$\mathscr{D}_N = \{1, 2, \ldots, N\},$$

$$\mathscr{F}_{i,M} = \{j;\ N+1 \leqslant j \leqslant N+M,\ a_{i,j} \neq 0\}, \qquad 1 \leqslant i \leqslant N,$$

$$\mathscr{C}_M = \bigcup_{i=1}^{N} \mathscr{F}_{i,M},$$

$$\mathscr{E}_M = \{N+1, N+2, \ldots, N+M\} \setminus \mathscr{C}_M.$$

In other words, $\mathscr{C}_M$ is the set of indices of the boundary points which are connected with some interior point in the sense that $a_{i,j} \neq 0$, and $\mathscr{E}_M$ is the set of indices of the boundary points which are not connected with any interior points. Typically (but not always) $\mathscr{E}_M$ is the set of indices of the corner points (see examples below).

We are now in a position to show the discrete maximum principles for (3) and (4).

THEOREM 4 (Strong discrete maximum principle). *Assume that*

$$a_{i,i} > 0, \quad a_{i,j} \leqslant 0, \qquad i \neq j, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+M,$$

$$\sum_{j=1}^{N+M} a_{i,j} \geqslant 0, \qquad 1 \leqslant i \leqslant N,$$

and that $A = (a_{i,j})$, $1 \leqslant i, j \leqslant N$, *is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{N+M})$ *satisfy*

$$\sum_{j=1}^{N+M} a_{i,j} w_j \leqslant 0, \qquad 1 \leqslant i \leqslant N.$$

If there exists some r $(1 \leqslant r \leqslant N)$ *such that*

$$\max_{1 \leqslant j \leqslant N+M} w_j = w_r \geqslant 0,$$

then,

$$w_j = w_r, \qquad j \in \mathscr{D}_N \cup \mathscr{C}_M,$$

$$w_j \leqslant w_r, \qquad j \in \mathscr{E}_M.$$

Proof. Since A is irreducible, for any k $(1 \leqslant k \leqslant N)$ there exists a sequence of nonzero elements of A of the form $\{a_{r,i_1}, a_{i_1,i_2}, \dots, a_{i_s,k}\}$ from Lemma 1. Thus we have

$$0 \geqslant \sum_{j=1}^{N+M} a_{r,j} w_j = a_{r,r} w_r + \sum_{\substack{j=1 \\ j \neq r}}^{N+M} a_{r,j} w_j$$

$$\geqslant \left(- \sum_{\substack{j=1 \\ j \neq r}}^{N+M} a_{r,j} \right) w_r + \sum_{\substack{j=1 \\ j \neq r}}^{N+M} a_{r,j} w_j = - \sum_{\substack{j=1 \\ j \neq r}}^{N+M} a_{r,j}(w_r - w_j)$$

$$= -a_{r,i_1}(w_r - w_{i_1}) - \sum_{j \in \mathscr{F}_{r,M}} a_{r,j}(w_r - w_j) - \sum_{\substack{j=1 \\ j \neq r \\ j \neq i_1}}^{N} a_{r,j}(w_r - w_j) \geqslant 0.$$

From the fact that $a_{r,i_1} < 0$, $a_{r,j} < 0$, $j \in \mathscr{F}_{r,M}$, we get

$$w_{i_1} = w_r,$$

$$w_j = w_r, \qquad j \in \mathscr{F}_{r,M}.$$

The same arguments yield

$$w_{i_1} = w_{i_2} = \cdots = w_{i_s} = w_k = w_r,$$

$$w_j = w_r, \qquad j \in \mathscr{F}_{i,M}, \quad i = i_1, i_2, \dots, i_s, k.$$

Since k $(1 \leqslant k \leqslant N)$ is arbitrary, we obtain

$$w_j = w_r, \qquad j \in \mathscr{D}_N \cup \mathscr{C}_M.$$

It is clear that

$$w_j \leqslant w_r, \qquad j \in \mathscr{E}_M.$$

Therefore, the proof is complete. ∎

REMARK 4. If $A=(a_{i,j})$, $1 \leqslant i, j \leqslant N$ is a matrix satisfying all conditions of Theorem 4, then $A^{-1}>0$ [19, p. 85].

REMARK 5. For $m=2$ and $\mathscr{L}=-\Delta$ (Δ the Laplacian), we consider a uniform mesh in both directions with mesh size h. For $P_i \in \Omega$, let $P_{i_1}-P_{i_4}$ be the four nearest points of P_i, where $P_i=(x_1, x_2)$, $P_{i_1}=(x_1+h_1, x_2)$, $P_{i_2}=(x_1, x_2+h_2)$, $P_{i_3}=(x_1-h_3, x_2)$, $P_{i_4}=(x_1, x_2-h_4)$, $0<h_j \leqslant h, 1 \leqslant j \leqslant 4$. The finite difference operator $-\Delta_h$ for the approximation to $-\Delta$ is defined by

$$-\Delta_h w_i \equiv -\frac{2w_{i_1}}{h_1(h_1+h_3)}-\frac{2w_{i_2}}{h_2(h_2+h_4)}-\frac{2w_{i_3}}{h_3(h_1+h_3)}-\frac{2w_{i_4}}{h_4(h_2+h_4)} + \left(\frac{2}{h_1h_3}+\frac{2}{h_2h_4}\right)w_i. \tag{5}$$

Then, the matrix $(a_{i,j})$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N+M$, associated with (5) satisfies all the conditions in Theorem 4 [3]. For the finite element method with piecewise linear polynomials, all the conditions in Theorem 4 are satisfied if all the angles of the triangles in the triangulation are less than or equal to $\pi/2$ [4].

REMARK 6. In [10, p. 448], the strong discrete maximum principle for $-\Delta_h$ with a rectangle in $\mathbf{R}^2$ is discussed.

THEOREM 5 (Weak discrete maximum principle [3]). *Assume that*

$$a_{i,i}>0, \quad a_{i,j} \leqslant 0, \qquad i \neq j, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+M,$$

$$\sum_{j=1}^{N+M} a_{i,j} \geqslant 0, \qquad 1 \leqslant i \leqslant N,$$

and that $A=(a_{i,j})$, $1 \leqslant i, j \leqslant N$, *is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{N+M})$ *satisfy*

$$\sum_{j=1}^{N+M} a_{i,j} w_j \leqslant 0, \qquad 1 \leqslant i \leqslant N.$$

Then,

$$\max_{1 \leqslant i \leqslant N+M} w_i \leqslant \max\left\{0, \max_{N+1 \leqslant j \leqslant N+M} w_j\right\}.$$

THEOREM 6. *Assume that*

$$a_{i,i} > 0, \quad a_{i,j} \leqslant 0, \qquad i \neq j, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+M,$$

$$\sum_{j=1}^{N+M} a_{i,j} \geqslant 0, \qquad 1 \leqslant i \leqslant N,$$

and that $A = (a_{i,j})$, $1 \leqslant i, j \leqslant N$ *is irreducibly diagonally dominant. Let* $(w_1, w_2, \ldots, w_{N+M})$ *satisfy*

$$\sum_{j=1}^{N+M} a_{i,j} w_j \leqslant 0, \qquad 1 \leqslant i \leqslant N.$$

If there exist some r $(N+1 \leqslant r \leqslant N+M)$ *and at least one* q $(1 \leqslant q \leqslant N)$ *such that*

$$\max_{1 \leqslant j \leqslant N+M} w_j = w_r \geqslant 0,$$

$$\sum_{j=1}^{N+M} a_{r,j} \geqslant 0, \qquad a_{r,j} \leqslant 0, \qquad j \neq r, \qquad 1 \leqslant j \leqslant N+M,$$

$$a_{r,q} < 0,$$

then,

$$\sum_{j=1}^{N+M} a_{r,j} w_j > 0$$

unless

$$w_j = w_r, \qquad j \in \mathscr{D}_N \cup \mathscr{C}_M,$$

$$w_j \leqslant w_r, \qquad j \in \mathscr{E}_M.$$

Proof. Assume that

$$\sum_{j=1}^{N+M} a_{r,j} w_j \leqslant 0.$$

Then, we have

$$\begin{aligned}
0 &\geqslant \sum_{j=1}^{N+M} a_{r,j} w_j = a_{r,r} w_r + \sum_{\substack{j=1\\ j\neq r}}^{N+M} a_{r,j} w_j \\
&\geqslant \left(- \sum_{\substack{j=1\\ j\neq r}}^{N+M} a_{r,j} \right) w_r + \sum_{\substack{j=1\\ j\neq r}}^{N+M} a_{r,j} w_j = - \sum_{\substack{j=1\\ j\neq r}}^{N+M} a_{r,j}(w_r - w_j) \\
&= - a_{r,q}(w_r - w_q) - \sum_{\substack{j=1\\ j\neq r\\ j\neq q}}^{N+M} a_{r,j}(w_r - w_j) \geqslant 0.
\end{aligned}$$

From the fact $a_{r,q} < 0$, it follows that

$$w_q = w_r = \max_{1 \leqslant j \leqslant N+M} w_j \geqslant 0.$$

Since $1 \leqslant q \leqslant N$, an application of Theorem 4 leads to

$$w_j = w_q = w_r, \qquad j \in \mathscr{D}_N \cup \mathscr{C}_M. \tag{6}$$

On the other hand, it is clear that

$$w_j \leqslant w_q = w_r, \qquad j \in \mathscr{E}_M. \tag{7}$$

Hence, (6) and (7) contradict the hypotheses in Theorem 6. Thus, we have

$$\sum_{j=1}^{N+M} a_{r,j} w_j > 0.$$

This completes the proof. ∎

In the sequel, we say that the boundary point P_r $(N+1 \leqslant r \leqslant N+M)$ has the *discrete interior point property* if there exists at least one q $(1 \leqslant q \leqslant N)$ such that $a_{r,q} \neq 0$.

REMARK 7. If $\sum_{j=1}^{N+M} a_{i,j} \geqslant 0$, $1 \leqslant i \leqslant N$, in Theorems 4–6 are replaced by $\sum_{j=1}^{N+M} a_{i,j} = 0$, $1 \leqslant i \leqslant N$, then $\max_{1 \leqslant j \leqslant N+M} w_j = w_r \geqslant 0$ in Theorems 4, 6, and $\max_{1 \leqslant i \leqslant N+M} w_i \leqslant \max\{0, \max_{N+1 \leqslant j \leqslant N+M} w_j\}$ in Theorem 5 are replaced by $\max_{1 \leqslant j \leqslant N+M} w_j = w_r$ and $\max_{1 \leqslant i \leqslant N+M} w_i \leqslant \max_{N+1 \leqslant j \leqslant N+M} w_j$, respectively.

REMARK 8. Replacing $(w_1, w_2, \ldots, w_{N+M})$ by $(-w_1, -w_2, \ldots, -w_{N+M})$ in Theorems 4–6, we obtain the *discrete minimum principles.*

3. SOME EXAMPLES

In this section, we give some examples to illustrate the discrete maximum principles. For a given domain Ω in $\mathbf{R}^2$ with smooth boundary Γ, P_1, P_2 denote the interior points of Ω and P_3-P_{12} denote the boundary points, as shown in Figure 1 with a uniform mesh $(h=1, N=2, M=10)$. We often use (5).

EXAMPLE 1.

$$-\Delta u + u = -5 \quad \text{in } \Omega,$$

$$u = -9 \quad \text{on } \Gamma.$$

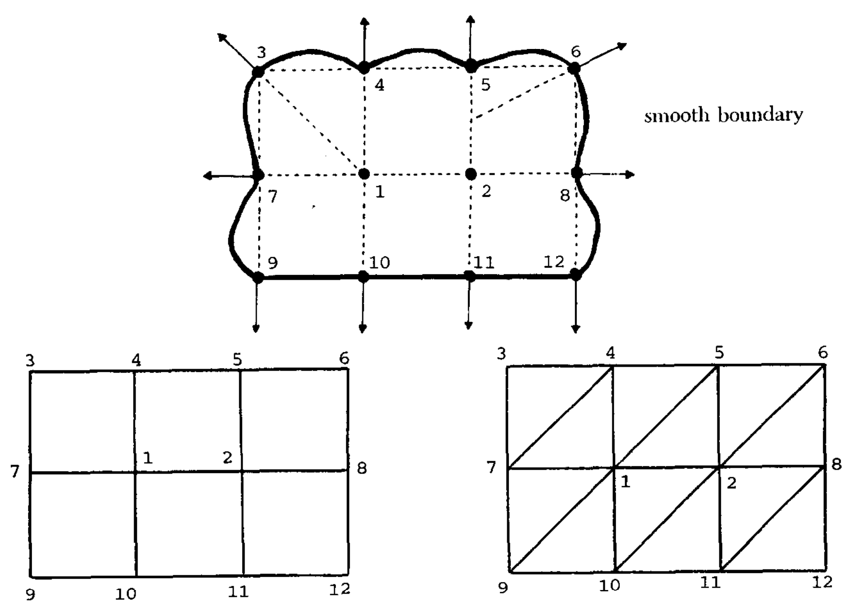

FIG. 1. Uniform mesh of the domain $(h=1,\ N=2,\ M=10)$: (a) domain and directions, (b) finite difference, (c) finite element.

The finite difference method with mesh points in Figure 1(b) leads to

$$\left[\begin{array}{rr|rrrrrrrrrr} 5 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 5 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\times \left[\begin{array}{r} -8 \\ -8 \\ \hline -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \\ -9 \end{array}\right] = \left[\begin{array}{r} -5 \\ -5 \end{array}\right].$$

In this case, $\mathscr{D}_N = \{1,2\}$, $\mathscr{C}_M = \{4,5,7,8,10,11\}$, $\mathscr{E}_M = \{3,6,9,12\}$. Since $\max_{1 \leqslant j \leqslant 12} w_j = w_1 = w_2 = -8$, the assumptions of Theorem 4 do not hold. However, note that those of Theorem 5 hold.

EXAMPLE 2.

$$-\Delta u = -3 \qquad \text{in } \Omega,$$
$$u = 1 \qquad \text{on } \Gamma.$$

The finite difference method with mesh points in Figure 1(b) leads to

$$\left[\begin{array}{rr|rrrrrrrrrr} 4 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ \hline -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{r} 0 \\ 0 \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} -3 \\ -3 \\ \hline \frac{1}{\sqrt{2}} \\ 1 \\ 1 \\ \frac{1}{\sqrt{5}} \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right].$$

The boundary matrix $(a_{i,j})$, $3 \leqslant i \leqslant 12$, $1 \leqslant j \leqslant 12$, denotes the first order approximation to $\partial/\partial \boldsymbol{\nu}$ on the boundary by means of linear interpolation [9, p. 37], where $\boldsymbol{\nu}$ is the direction in Figure 1(a). On the points $P_3 - P_5$, $P_6 - P_8$, P_{10}, P_{11}, which have the discrete interior point property, the conclusions in Theorem 6 hold. However, on the points P_9, P_{12}, which have not the discrete interior point property, Theorem 6 is not valid.

EXAMPLE 3.

$$
\begin{aligned}
-\Delta u &= 0 \qquad \text{in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} &= g \qquad \text{on } \Gamma.
\end{aligned}
$$

The finite element method by piecewise linear polynomials with mesh points in Figure 1(c) leads to

$$
\left[\begin{array}{rr|rrrrrrrrrr}
4 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
-1 & 4 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 1 & -\frac12 & 0 & 0 & -\frac12 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -\frac12 & 2 & -\frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -\frac12 & 2 & -\frac12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac12 & 1 & 0 & -\frac12 & 0 & 0 & 0 & 0 \\
-1 & 0 & -\frac12 & 0 & 0 & 0 & 2 & 0 & -\frac12 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -\frac12 & 0 & 2 & 0 & 0 & 0 & -\frac12 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 0 & 1 & -\frac12 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 2 & -\frac12 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 2 & -\frac12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 0 & 0 & -\frac12 & 1
\end{array}\right]
\left[\begin{array}{r}
0 \\ 0 \\ \hline 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ -4 \\ 2
\end{array}\right]
=
\left[\begin{array}{r}
0 \\ 0 \\ \hline 0 \\ 2 \\ 2 \\ 0 \\ 2 \\ 2 \\ 3 \\ -7 \\ -7 \\ 3
\end{array}\right].
$$

In this case, the boundary matrix $(a_{i,j})$, $3 \leqslant i \leqslant 12$, $1 \leqslant j \leqslant 12$, denotes the approximation to $\partial/\partial \mathbf{n}$ as the natural boundary condition with an application of Green's formula. On the points P_4, P_5, P_7, P_8, which have the discrete interior point property, the conclusions of Theorem 6 are valid. However, on the points P_3, P_6, P_9, P_{12}, which have not the discrete interior point property, we cannot determine whether or not Theorem 6 holds. It is noted that on the

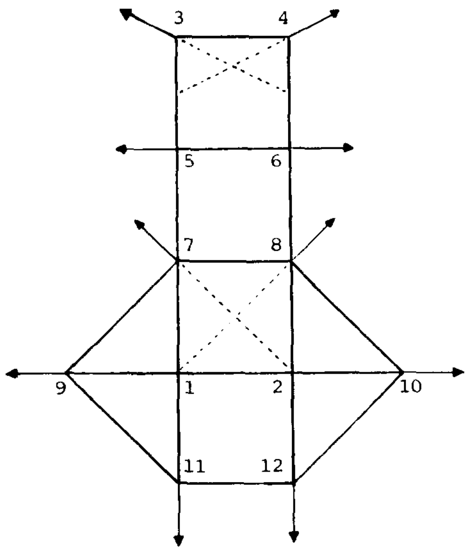

FIG. 2. Mesh points and directions ($h=1$, $N=2$, $M=10$).

points P_{10}, P_{11}, which have the discrete interior point property, the discrete minimum principles are valid.

EXAMPLE 4. We consider the following equations associated with the finite difference method with mesh points and directions as in Figure 2:

$$\left[\begin{array}{cc|cccccccccc} 4 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 4 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

In this case, $\mathscr{D}_N = \{1,2\}$, $\mathscr{C}_M = \{7,8,9,10,11,12\}$, $\mathscr{E}_M = \{3,4,5,6\}$, and

$$w_j = 1, \qquad j \in \mathscr{D}_N \cup \mathscr{C}_M,$$
$$w_j \leqslant 1, \qquad j \in \mathscr{E}_M.$$

However, on the points P_3, P_4, which have not the discrete interior point property, we have

$$\sum_{j=1}^{12} a_{3,j} w_j > 0, \qquad \sum_{j=1}^{12} a_{4,j} w_j < 0.$$

4. APPLICATION

Let Ω be the circular domain in $\mathbf{R}^2$ given by

$$\Omega = \left\{ (x_1, x_2) \in \mathbf{R}^2;\ x_1^2 + x_2^2 < 1 \right\}.$$

We consider the radiation cooling problem with the nonlinear boundary condition obeying Newton's law of cooling:

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= g(x, u) && \text{on } \Gamma. \end{aligned} \tag{8}$$

Here $u(x)$ represents the absolute temperature, so that $u(x)$ is required to be *positive*. We assume that

(H-1) $g(x, u)$ is twice continuously differentiable in $\overline{\Omega}$ for all u,
(H-2) $g(x, 0) > 0$, $g(x, 1) = 0$,
(H-3) $g_u(x, u) \equiv \partial g / \partial u < 0$ in $\overline{\Omega}$ for $u > 0$,
(H-4) $g_{uu}(x, u) \equiv \partial^2 g / \partial u^2 < 0$ in $\overline{\Omega}$ for $u > 0$.

Under these hypotheses, Cohen [5] established the uniqueness and existence of the positive solution of (8).

For the discrete problem of (8), mesh points are constructed, for example, as shown in Figure 3. By using the finite difference method and the first order approximation to $\partial / \partial \mathbf{n}$, we have the following nonlinear algebraic

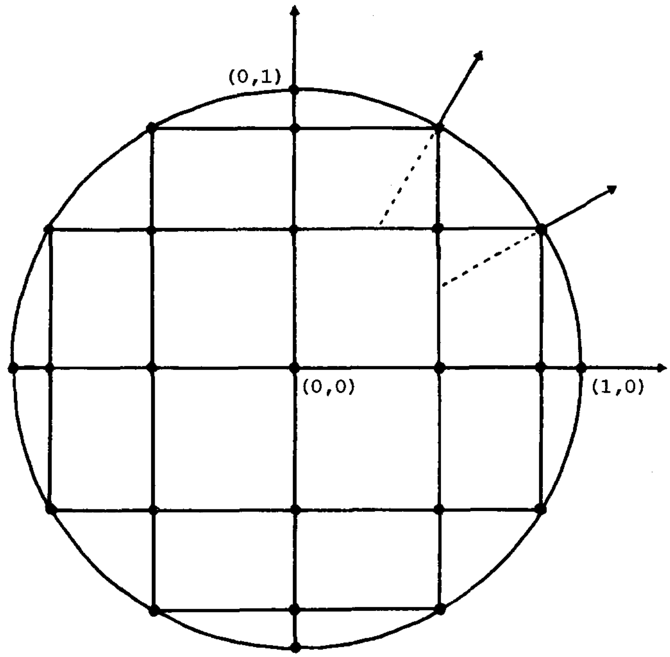

FIG. 3. Circular domain and mesh points.

equations:

$$\sum_{j=1}^{N+M} a_{i,j} w_j = \begin{cases} 0, & 1 \leqslant i \leqslant N, \\ g(p_i, w_i), & N+1 \leqslant i \leqslant N+M. \end{cases} \tag{9}$$

We note that the coefficients $a_{i,j}$, $1 \leqslant i, j \leqslant N+M$, satisfy all conditions of Theorem 6 and $\mathscr{E}_M$ is the empty set. Also all the boundary points have the discrete interior point property and

$$\sum_{j=1}^{N+M} a_{i,j} > 0, \qquad 1 \leqslant i \leqslant N. \tag{10}$$

Then, we can prove the uniqueness of the positive solution $w_i > 0$, $1 \leqslant i \leqslant N + M$, of (9) by applying the discrete maximum principles.

Proof of the uniqueness of the positive solution of (9). Let $(w_1, w_2, \ldots, w_{N+M}), (y_1, y_2, \ldots, y_{N+M})$ be the two positive solutions of (9). By putting $z_i = w_i - y_i$, $1 \leqslant i \leqslant N+M$, we have

$$\sum_{j=1}^{N+M} a_{i,j} z_j = 0, \qquad 1 \leqslant i \leqslant N, \tag{11}$$

$$= g(p_i, w_i) - g(p_i, y_i), \qquad N+1 \leqslant i \leqslant N+M. \tag{12}$$

Assume that there exists some i_0 $(1 \leqslant i_0 \leqslant N+M)$ such that $z_{i_0} > 0$ (or $z_{i_0} < 0$).

Consider the case $z_{i_0} > 0$. Thus, $\max_{1 \leqslant j \leqslant N+M} z_j = z_r > 0$. If $1 \leqslant r \leqslant N$, then an application of Theorem 4 leads to

$$z_j = z_r > 0, \qquad 1 \leqslant j \leqslant N+M.$$

However, this solution cannot satisfy (11), because of (10). Hence, we have $N+1 \leqslant r \leqslant N+M$. By applying Theorem 6, we get

$$\sum_{j=1}^{N+M} a_{r,j} z_j > 0. \tag{13}$$

On the other hand, from (12) and (H-3), it follows that

$$\sum_{j=1}^{N+M} a_{r,j} z_j = g(P_r, w_r) - g(P_r, y_r) = g_u(P_r, \xi) z_r < 0 \qquad \text{with} \quad y_r < \xi < w_r.$$

This is a contradiction to (13).

In the case $z_{i_0} < 0$, we have a similar contradiction. Therefore, we obtain $z_i = 0$, $1 \leqslant i \leqslant N+M$. This implies the uniqueness of the positive solution. Thus, the proof is complete. ∎

For the finite element approximation, we may refer to [14].

5. CONCLUDING REMARKS

In the discrete matrix equations associated with the elliptic boundary value problem, it is desirable to reduce the number of the boundary points belonging to $\mathscr{E}_M$ as much as possible, in order to obtain good numerical solutions from the viewpoint of the maximum principles. The most desirable feature is that not only is $\mathscr{E}_M$ the empty set, but also all the boundary points have the discrete interior point property. In particular, for the finite element approximation with piecewise linear polynomials and lumping operator, this property is easily satisfied by the appropriate triangulation of nonnegative type [4].

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REFERENCES

1 J. H. Bramble and B. E. Hubbard, On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation, *Numer. Math.* 4:313–327 (1962).

2 J. H. Bramble and B. E. Hubbard, A theorem on error estimation for finite difference analogues of the Dirichlet problem for elliptic equations, in *Contributions to Differential Equations*, Vol. 2, Wiley, New York, 1963, pp. 319–340.

3 P. G. Ciarlet, Discrete maximum principle for finite-difference operators, *Aequationes Math.* 4:338–352 (1970).

4 P. G. Ciarlet and P. A. Raviart, Maximum principle and uniform convergence for the finite element method, *Comput. Methods Appl. Mech. Engrg.* 2:17–31 (1973).

5 D. S. Cohen, Generalized radiation cooling of a convex solid, *J. Math. Anal. Appl.* 35:503–511 (1971).

6 L. Collatz, *The Numerical Treatment of Differential Equations*, 3rd ed., Springer, New York, 1966.

7 K. O. Friedrichs and H. B. Keller, A finite difference scheme for generalized Neumann problems, in *Numerical Solution of Partial Differential Equations* (J. H. Bramble, Ed.), Academic, New York, 1966, pp. 1–19.

8 D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1983.

9 D. Greenspan, *Introductory Numerical Analysis of Elliptic Boundary Value Problems*, Harper and Row, New York, 1965.

10 E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, Wiley, New York, 1966.

11 K. Ishihara, On finite element schemes of the Dirichlet problem for a system of nonlinear elliptic equations, *Numer. Funct. Anal. Optim.* 3:105–136 (1981).

12 K. Ishihara, Finite element approximations applied to the nonlinear boundary value problem $\Delta u = bu^2$, *Publ. Res. Inst. Math. Sci.* 18:17–34 (1982).

13 K. Ishihara, Monotone explicit iterations of the finite element approximations for the nonlinear boundary value problem, *Numer. Math.* 43:419–437 (1984).

14 K. Ishihara, Finite element solutions for radiation cooling problems with nonlinear boundary conditions, *RAIRO Modélisation Mathématique et Analyse Numérique* 20:461–477 (1986).

15 J. M. Ortega, *Numerical Analysis*, Academic, New York, 1972.
16 M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.
17 R. Sperb, *Maximum Principles and Their Applications*, Academic, New York, 1981.
18 G. Stoyan, On a maximum principle for matrices, and on conservation of monotonicity, with applications to discretizaton methods, *Z. Angew. Math. Mech.* 62:375–381 (1982).
19 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

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